Muʿāmalāt and otherwise in the *Liber mahamaleth*

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Abstract

The twelfth-century Iberian *Liber mahamaleth* was discovered and described by Jacques Sesiano in 1986; in 2010, a critical edition of the work was produced by Anne-Marie Vlasschaert. Both agreed that the title of the work reflects Arabic *muʿāmalāt*, "[the mathematics of] social intercourse"; that the work goes beyond *muʿāmalāt* mathematics by integrating its material with proofs in Euclidean style; and that it is an independent creative compilation, not a translation of a single work. Charles Burnett has suggested the compiler-author to be Gundisalvi.

The present paper delineates the development of the notion of *muʿāmalāt* as a branch of practical arithmetic from the early ninth through the mid-twelfth century and locates the contents of the *Liber mahamaleth* with more precision in respect to it, using also Castilian and related early Italian abbacus material as well as Gundisalvi's *De divisione philosophiae*. Analysis of that aspect of the text that clearly falls outside the *muʿāmalāt* tradition leads to the conclusion that the *Liber mahamaleth* is a translation of what Gundisalvi speaks of as "the book which in Arabic is called *Mahamalech*", and that the integration of *muʿāmalāt* material with the techniques of theoretical mathematics was thus a product of al-Andalus culture and not of the Latin translation movement.

In the end two other pieces of sophisticated theoretical arithmetic known only from Latin and Romance vernacular sources – a systematic scrutiny of certain properties of the Nicomachean means and an examination of a particular type of complex series – are shown also to be plausible products of that phase of al-Andalus learned culture where it influenced Hebrew and Latin much more than later Arabic learning.

At the first Maghreb Symposium on the History of Arabic Mathematics in 1986, Jacques Sesiano revealed his discovery of the *Liber mahamaleth* and presented an analysis of the work (published in [1988]). The publication announced an edition, which however never appeared; but in a number of other publications he has referred to the treatise.[1]

Instead, a preliminary edition of the work was contained in Anne-Marie Vlasschaert's PhD thesis (2002-2003), while a completed version from her hand appeared in [2010].

Sesiano and Vlasschaert agree that the title refers to the Arabic concept *muʿāmalāt*, "[the mathematics of] social intercourse",[2] and that it represents either a translation or a calque of Arabic *Kitāb al-muʿāmalāt*. Both also agree that the creator of the Latin text is either a creative compiler or to be characterized as a genuine author (relying of course on previous material, in the likeness of almost any mathematical author in history).

In [1988: 70], Sesiano thought the treatise was written in Castile during the second half of the twelfth century by an author "about whose name it is not even possible to formulate conjectures".[3] In [1993: 215] and [2000: 71] he identified the author as Johannes Hispalensis, without presenting his arguments,[4] which in [2001: 10] becomes "composed no doubt by Johannes Hispalensis (Jean de Seville) around 1240 in Seville". He still does not explain the origin of this conviction, but it is probably related to what is told in [Burnett, Zhao & Lampe 2007: 145]: that the *Liber mahamaleth* "shares passages both with the *De divisione philosophiae* of the archdeacon

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[1] In [Sesiano 1987], about the "slanted ladder" problems; in [Sesiano 1993: 315], a passing reference; and in [2001: 10–13], a discussion of some of the algebraic problems solved in the work, and an observation about the absence of a perceptible impact; maybe also elsewhere.

[2] Except when specified below, I disregard the much broader use of the term in the legal-religious tradition accounted for in [Bernand 1993], which in any case is not directly relevant to the *Liber mahamaleth*.

[3] Trans. JH – as all translations in the following with no identified translator.

[4] I may of course have overlooked an intermediate publication from his hand giving the arguments, for which I shall then apologize.

resident at Toledo cathedral, Dominicus Gundissalinus (d. after 1180) and with the Toledan *regule*" ascribed, precisely, to Johannes Hispalensis [Boncompagni 1857b:25; Burnett 2002: 64] and probably written around 1147. On the same basis, however, Burnett [2002], points to Gundisalvi as the likely author; Vlasschaert [2010: I27–30] tends to follow him. This would return the date to Sesiano's original suggestion, which was based on internal evidence. It would also fit Gundisalvi's very free use of Arabic material in independent writings such as his *De divisione philosophiae* [ed. Baur 1903] (and even his far from faithful translation style as evident in his version of al-Fārābī's *Catalogue* [ed. Palencia 1953: 87–115]).

The developing range of *muʿāmalāt* mathematics

It is quite obvious that much of the contents of the *Liber mahamaleth* does come from the *muʿāmalāt*-tradition, while another part comes from sources that have little to do with "[the mathematics of] social intercourse" – not least the copiously present demonstrations and the use of techniques borrowed from the *Elements* (with reference to book and proposition), but also the metatheoretical introduction about the nature of number. This is discussed by both Sesiano and Vlasschaert, and I shall add nothing on that account. However, as both recognize, their considerations do not exhaust the question about the sources of the work and its relations to Arabic *muʿāmalāt*, in particular because contemporary writings on similar matters from al-Andalus have not survived, or in any case have not been identified. On the other hand, comparison with other sources will allow us to gain more insight or at least to locate the open questions with more precision.

The question inherent in the phrase "*muʿāmalāt* and otherwise" presupposes that *muʿāmalāt* is something well-defined. This is likely to a mistake – cf. also [Vlasschaert 2010: I9 n.9]. It seems obvious, however, that *muʿāmalāt* when used about a particular type of mathematics is a category impressed from outside, not a notion used by its practitioners. Merchant's and their ilk are not likely to have thought of their calculations as "[the mathematics of] social intercourse" but simply as *ḥisāb*, "computation", if at all characterizing them – in a Norwegian merchants' "mirror" from c. 1195, all the father has to say about calculation when admonishing his son is “practise [*gerðu* / “do”] number skill [*tölvisan*] well, that is much needed by merchants” [ed. Keyser et al 1848: 7]. *Muʿāmalāt* in this sense is almost certainly a scholars’ category,[5] as reflected in its very first known appearance: in al-Khwārizmī’s *Algebra* [ed. Rosen 1831: $_A$48; Rashed 2007: 97]. Here, *muʿāmalāt* stands for the rule of three and nothing but that; in particular, inheritance calculations are not included but treated later in the work. It is likely but not certain that the term was taken over and twisted by al-Khwārizmī himself when he found himself in need of an adequate chapter heading: in eighth-century Iraq, the term *muʿāmalah* seems to have been used mostly about “the arithmetical determination of transactions and credit operations concerning the cultivations of the Arabic regions inhabited by the *umma*” and about subterfuges by which interest could be disguised as legal business [Bernand 1993: 256] – topics that clearly go beyond the rule of three and are much more juridically specific.[6]

The tenth and the early eleventh century gives us more information about what was (by then, not necessarily in the early ninth century) understood under *muʿāmalāt*.

Al-Fārābī, in *Iḥṣāʾ al-ʿulūm* [ed. trans. Palencia 1953: $_A$54, 39] speaks of “*muʿāmalāt* of cities and market places” under practical arithmetic and says nothing more. The Brethren of Purity, in their *Letters*, list “the sciences of *ḥisāb* and *muʿāmalāt*” just after the sciences “of language and grammar” and just before those of “poetry and prosody” in a list of sciences based

5 “Scholars” in this sense fall into two groups: those who also wrote about astronomical matters and are characterized as “mathematicians” by modern historians of science, and legal scholars writing about mathematical matters that might be of judicial interest.

6 It would be easier to be sure about al-Khwārizmī’s creative role if we possessed at least a date for ibn Turk’s work about *muʿāmalāt*; but all al-Nadīm gives in the *Fihrist* [trans. Dodge 1970: 664; Suter 1900: 18 n.a] is the title, and an extremely vague suggestion that it may be a short work (it is not told to be subdivided but stands as the neighbour of a compilation about the totality of *ḥisāb* which is said to be divided into “six sections”). The title may even be spurious – Suter suspects ibn Turk’s *muʿāmalāt* item to be a contamination taken over from the ascription of a work with the same title to his grandson a few lines later.

on "the pursuit of a living and a just life in the present world" [Rebstock 1992: 13]. Later in the century, al-Āmirī offers somewhat more precision, if not about the mathematics involved then at least about which kinds of *transactions* are implied – namely all of those that a judge might get involved in: sale and lease, marriage and divorce, complaint and legal proof, deposits and loans, and legacies and inheritance [Rebstock 1992: 27]. Like al-Fārābī and the Brethren he counts this as a branch of mathematics. None of them suggests anything beyond the strictly utilitarian. Nor do they necessarily *rule out* interest in "supra-utilitarian" matters,[7] but explicit interest in proof seems to be excluded by the listing under practical as opposed to theoretical arithmetic.

We learn details from ibn al-Haytham's *Treatise on muʿāmalāt-calculation* [ed., trans. Rebstock 1996], which contains nothing supra-utilitarian. It starts by a long explanation of numerical calculation, with much space allotted to fractions (including fractions of fractions and ascending continued fractions such as $\frac{1}{10}+\frac{1}{6}\cdot\frac{1}{10}$), then gives a brief exposition of the rule of three – as normal among Arabic mathematicians explained as a proportional relation between four numbers of which three are known and one is unknown. Finally this rule (mostly in the shape where one of the numbers is one) is applied to a few examples: proportional division of a harvest; transformation of goods into monetary value; land taxation; conversion of gold weight into monetary units; wages; palm tax; and fodder for animals.

On the other hand Abū Kāmil, when submitting the supra-utilitarian problem of the "hundred fowls" to theoretical analysis [ed. Suter 1910] speaks of it as a kind of *ḥisāb,* belonging more specifically to the "rarities" (*tarāʾif*) of this field, and does not refer at all to the *muʿāmalāt*.

So, until the early eleventh century, *muʿāmalāt* calculation seems to concern the transactions really occurring in social life – but already much more than the mere rule of three.

[7] That is, problems that according to their topic seem to deal with practical life but which present a mathematical structure or a degree of complexity that would never occur in practice. Most "recreational problems" belong to this category.

Muʿāmalāt in al-Andalus

This is a far cry from the *Liber mahamaleth*, which not only dedicates much attention to Euclidean proofs but also abounds in supra-utilitarian questions. However, it is quite possible that the term *muʿāmalāt* was understood differently in al-Andalus in the twelfth century, as suggested by Vlasschaert [2010: ₁9 n.9].[8] Unfortunately, none of the treatises that were written about the topic in the Maghreb (in the original sense encompassing al-Andalus) have survived [Djebbar 2001: 328]. However, we have one description of the field, or rather of the contents of a book dealing with it. In Gundisalvi's *De divisione philosophiae*, the branches of practical arithmetic are listed: purchase and sale, barter, lease, payment and saving, measurement of depths and widths and heights and other extensions of things. "All these are sufficiently dealt with in the book which in Arabic is called *Mahamalech*" [ed. Baur 1903: 93].[9]

Ahmed Djebbar, in order to know about al-Andalus *muʿāmalāt*, suggests that we look at the *Liber mahamaleth*, but for the present question this would lead nowhere but into a vicious circle. So, what independent sources do we have for the character of Ibero-Islamic practical arithmetic?

We have none that characterize themselves in a way that suggests *muʿāmalāt*-calculation, so we have to be satisfied with what might reflect the practice of *ḥisāb* in al-Andalus. What Gundisalvi says about "the book which in Arabic is called *Mahamalech*" gives firm support to this approach.

Unfortunately, even books about *ḥisāb* in Arabic from al-Andalus have not survived, nor anything in Romance languages from the twelfth or thirteenth centuries. But we have two Castilian works from the fourteenth

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[8] And not only in al-Andalus. In ibn Thabāt's *Ghunyat al-Ḥussāb* ("Treasures of the Calculators"), written around 1200 CE in Baghdad, the *muʿāmalāt*-chapter [ed., trans. Rebstock 1993: 43–80] includes the supra-utilitarian "rarities" (here *nawādir*) belonging to the domain. The larger range of the concept seems to reflect the passage of time, not the split East / West.

[9] Since all manuscripts used by Ludwig Baur for the edition go back to a single "probably already secondary and error-ridden" archetype [Baur 1903: 154], it is quite possible that the original had *Liber mahamaleth*. Fortunately, however, the spelling with *c* allows us to distinguish this Arabic treatise from the Latin compilation.

and fifteenth century and evidence that at least one of them has its roots in the Iberian first half of the thirteenth century or earlier, *not* in the Italian abbacus environment (which would make it irrelevant for the present purpose).

One of them is a small fifteenth-century treatise *De arismetica*, described with texts excerpts in Caunedo del Potro 2010.[10] It is hardly in pure Iberian tradition – that would be astonishing in view of the late date; but as far as its contents is concerned it might well be, to judge from Caunedo del Potro's description and excerpts. What can be seen from Caunedo del Potro's analysis is, firstly, that the manipulation of fractions is by far the most important technique; secondly, that the topics dealt with are just as bound to real social life as in ibn al-Haytham's treatise.[11] This agrees with the natural expectation that *muʿāmalāt* in the restricted tenth-century style was still part of the Iberian tradition, but is not particularly informative.

The other treatise teaches us more. It is a *Libro de arismética que es dicho alguarismo*, "Book about Arithmetic That is Called Algorism", written in 1393. It is known from a sixteenth-century copy and builds on material from no later than the early fourteenth century. It was edited by Caunedo del Potro in [Caunedo del Potro & Córdoba de la Llave 2000].

In agreement with its being characterized as an "algorism", the treatise starts by introducing the Hindu-Arabic numerals and the appurtenant computational techniques. Next come, not very orderly, problems, many of them of the classical supra-utilitarian types, mixed up with various rules. Particularly striking is the formulation of the rule of three as a "rule if so

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[10] Actually, Betsabé Caunedo del Potro does not tell the precise age of the manuscript, which may be uncertain, discussing it merely in the context of the fifteenth century; but the orthography fits the fourteenth or fifteenth century, and the manuscript is bound together with another one dated to around 1450 [Caunedo del Potro 2004: 41 n.30]

[11] According to Caunedo del Potro, 2 out of 48 problems are treated by means of "operaciones elementales", 4 by means of "proporciones" (probably the rule of three or one of its equivalents), and 42 by means of manipulation of fractions. The topics dealt with are numerical computation (20 problems); mental computation (2); prices of goods (16); distribution of money (2); alloying (2); interest (5); and equivalence of coins (1).

much were so much, what would so much be" (p. 146, referred to under this name on pp. 181, 183, and 188). The rule prescribes to arrange the three known numbers sequentially and to multiply then the second by the third, dividing finally by the first. This linear organization is strikingly different from that ordering in a rectangular scheme which is used in Fibonacci's *Liber abbaci* and which is also reflected in the mistranslation of *mubāyin* (meaning "different [in kind]") as "opposite" by both Gherardo [ed. Hughes 1986: 255] and Robert of Chester [ed. Hughes 1989: 64] in their translations of al-Khwārizmī's treatment of the topic. In contrast, it correspond perfectly to the Sanskrit versions of the rule – but the idea is too close at hand and the distance too large for this coincidence to serve as evidence of diffusion, in particular in the absence of known intermediaries.

There are also a number of rules for the arithmetic of mixed numbers and composite fractions – for instance p. 182, "if you want to multiply integer and fraction and fraction of fraction in integer and fraction of fraction".[12] Clearly, this idea comes from the Arabic use of ascending continued fractions. However, while Fibonacci's presentation of these never caught on (for which reason we can be sure that the concept was not reimported from Italy), here they are put to good use, as a way to deal with units, subunits and sub-sub-units – it is immediately explained that "this is as if they said to you to multiply 5 quintals of iron and three arrobas and 6 pounds at 25 maravedi, 7 dinars and 4 medias the quintal".[13]

The inference that the *Libro ... dicho alguarismo* represents a direct continuation of an Iberian tradition and not a mere import from Italy is supported by evidence that commercial arithmetic was written about in Castilian at least before 1228 and probably before 1202 (the dates of the second and the first edition of the *Liber abbaci*, respectively). This evidence is found in the thirteenth-century Vatican manuscript Palat. 1343, one of the earliest manuscripts of the *Liber abbaci*.[14] On fol. 47$^{r–v}$ (new foliation)

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[12] "Sy quisyeres multiplicar sano e roto e roto de roto en sano e roto de roto".

[13] "como sy te dixiesen que multiplicases 5 quintales de fierro e 3 arrovas e 6 libras a 25 maravedís a 7 dineros e 4 medias el quintal"

[14] The text is incomplete, which is the likely reason Boncompagni did not use it

Fibonacci here tells to have used a work by a "Castilian master" for his chapter on barter (cf. [Boncompagni 1851: 32]).

... and in the *Liber mahamaleth*

We may now return to the *Liber mahamaleth*. Firstly, the abundance of supra-utilitarian problems in the *Libro ... dicho alguarismo* makes it likely that these had become part of the *muʿāmalāt* tradition in al-Andalus, and that the abundant presence of the genre in the *Liber mahamaleth* thus really corresponds to what was habitual in the field it claims to present. Secondly, the presence of ascending continued fractions in the *Libro ... dicho alguarismo* confirms that the "integers with fractions and fractions of fractions" (for instance, [ed. Vlasschaert $_{II}$2010: 33]) can also confidently be seen as a continuation of their use in al-Andalus *muʿāmalāt* (which is no surprise).

Thirdly, it throws light on the presentation of the rule of three in the *Liber mahamaleth*, which is far from straightforward. At first it speaks of the situation where we have "four proportional numbers of which three are known and one is unknown" [ed. Vlasschaert 2010: $_{II}$185], which is quite in the style of numerous Arabic expositions (cf. above on ibn al-Haytham). Of these four numbers the first and the fourth are declared "partners" (*socii*), and so are the second and the third. The partner of the unknown number is then to divide one of the others, and the outcome to be multiplied by the partner of the dividend; or the two known partners are to be multiplied, and the outcome divided by the partner of the unknown number (this, and not the variant where division comes first, is the rule of three). The same formulation is found in the "Toledan *regule*" [ed. Burnett, Zhao & Lampe 2007: 155], which as mentioned shares other formulations with the *Liber mahamaleth*, but as far as I know it turns up nowhere else in the world.

However, on the next page of the edition, under the heading "Buying and selling", we find a different formulation: multiply the middle in the last, and divide the product by the first. This presupposes the same linear arrangement as the *Libro ... dicho alguarismo*, which must then be assumed to belong to the local *muʿāmalāt* legacy.

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as the basis for his edition of the work.

And then, on p. 216, we find another rule, "you should always multiply with the thing that is of a different kind, neither the profit by the profit nor the capital by the capital but the profit by the capital and conversely". This corresponds to al-Khwārizmī's reference to what is *mubāyin*, and also to what is said in Sanskrit treatises from Bhāskara I onward and in many Arabic presentations of the rule – and not least to the standard formulation of the rule in Italian abbacus treatises.[15]

The question of algebra in *muʿāmalāt*

This leaves us with two major possible kinds of non-*muʿāmalāt* constituents of the *Liber mahamaleth*: algebra, and the many Euclidean demonstrations. The latter, as said above, have been discussed by Sesiano as well as Vlasschaert. They are clearly not borrowed from the *muʿāmalāt*-tradition but a tool *applied* by the compiler-author to his borrowed material. Let us therefore turn to algebra.

Copious blind cross-references leave no doubt that the original version of the *Liber mahamaleth* contained an orderly presentation of algebra, and correct cross-references to Abū Kāmil's algebra demonstrate that the compiler-author was familiar with that work. Moreover, the observation on p. 427 that an indeterminate problem is to be solved according to the procedures of algebra but not according to Abū Kāmil shows that the two are not identical. Since al-Khwārizmī's algebra does not deal with indeterminate problems, this already seems to rule out that the orderly presentation of the field was copied from al-Khwārizmī. This inference is confirmed on p. 209, where a solution *secundum algebra* makes use of two unknowns, *res* and *dragma*, quite in the style of Arabic algebra from the later ninth century onward (Abū Kāmil makes use of the technique in his discussion of the problem of the "hundred fowls" without suggesting

[15] Since this passage is only found in one manuscript, it could be suspected to be interpolated. However, the manuscript in question is by far the earliest one (late twelfth or possibly very early thirteenth century, Italian but made from a Toledan model) [Vlasschaert 2010: $_{I}$42]. The copyist seems moreover not to have been mathematically competent (*id.* p. 44), which should rule out meaningful interpolations.

it to be his own invention [ed. trans. Suter 1910]); nothing similar, however, can be found in al-Khwārizmī.

So, the algebra of the *Liber mahamaleth* is derived directly from Arabic algebra, not from some translation into Latin (Abū Kāmil's algebra was translated only in the fourteenth century, see [Sesiano 1993: 315]). But does it belong to the *muʿāmalāt*-tradition?

We have little evidence from elsewhere of algebra being integrated into *al-muʿāmalāt*, or just into *hisāb*. The exceptions I am aware of are

– Al-Karajī's *Kāfī* and *Fakhrī*. In the *Kāfī* [ed., trans. Hochheim 1878: III, 14–27], a number of "noteworthy problems" are solved by means of algebra, most of them dealing with topics from the (enlarged) *muʿāmalāt* area and the domain of practical geometry. Among the topics are wages for a worker; "give-and-take"; "purchase of a horse"; alloying; partnership; taxation; payment in mixed coin. Most lead to first-degree equations; in the end al-Karajī says that they have been taken from various authors having a particular predilection for this genre.[16] Woepcke's paraphrase of the *Fakhrī* [1853] unfortunately does not always specify which topic hides behind his numerical equations,[17] but we find at least "give-and-take", alloying and wages (some of them strict parallels to problems from the *Kāfī*). Besides that we find courier problems (p. 82) with one courier moving with constant and the other with arithmetically increasing speed, either starting at the same moment or the latter with delay; if delay is involved, this leads to a three-term quadratic equation. Several problems (p. 84, about wages and about the prices of pieces of cloth) involve square roots of amounts of money.

[16] Since al-Karajī excuses not to have praised and thanked these predecessors, we may perhaps presume that he has borrowed not only the problems but also the procedures by which they are solved. All the writings in question have been lost, but if (*if!*) their procedures were really algebraic it would fit the impression given by al-Karajī's own algebraic terminology that he draws on a tradition that is independent of al-Khwārizmī – cf. [Høyrup 2001: 117 n.50].

[17] For instance, a piece of literal translation and Arabic text on p. 139 reveals that a problem expressed in pure equations in *x*, *y* and *z* on p. 90 is actually a give-and-take problem involving three men, and that it is solved by means of two unknowns, "thing" and *dirham* (Woepcke translates the latter as "measure").

– Bahāʾ al-Dīn al-ʿĀmilī's *Khulāṣah al-ḥisāb* ("The Summary of Algebra") from c. 1600, whose chapter 8 presents the names of algebraic powers together with the rules for the six basic cases, each provided with an example, while the final chapter 10 lists nine problems "solved by varying methods"; for six of these, one of the methods is simple first-degree algebra, in one case (the pond-variant of the ladder-problem) very simple second-degree algebra.
– Ibn Badr's *Ikhtiṣār al-jabr wa'l-muqābalah*, written before 1344 (and after Abū Kāmil's times), perhaps in al-Andalus [ed., trans. Sánchez Pérez 1916][18] – like the *Fakhrī* an example of *ḥisāb* being integrated into algebra rather than the reverse (the title being, adequately, *algebra*). Here, algebraic techniques are applied to a number of (mostly supra-utilitarian) *ḥisāb*-problems dealing with relatively determined gain and absolutely defined alms; dowries; wheat and barley; soldiers taking booty; couriers of which one goes with constant and the other with arithmetically increasing speed or with constant speed but delay; "give-and-take" between two or three possessions; purchase of cloth (with the structure of a two-participant "purchase of a horse"); and geese eating and dying off. Most are of the first degree, but in one of the dowry problems a second dowry equals the square root of the first one, and in another one it is equal to the root of the first dowry diminished by one; the courier problems involving arithmetically increasing speed are inherently of the second degree, but they lack the constant term that arises in problems involving a delay. Several solutions make use of two variables called *thing* and *dinar*.

Several of the supra-utilitarian problem types treated by ibn Badr are also found in the *Libro ... dicho alguarismo*. At the general level this is not strange, similar problems are found "everywhere" – also in al-Karajī, as we have seen. Some details, however, are suggestive. Courier problems involving arithmetical increase but without delay are found on pp. 163

[18] Ibn Badr's treatise is known from a copy contained in a manuscript possessed by the Escorial Library and written in an apparently Iberian hand in 1344 [Sánchez Pérez 1916: xvi] – at least circumstantial evidence that ibn Badr's work was known in al-Andalus if not composed there.

and 169 (3 problems in total);[19] in all cases the solution is given according to an unexplained rule, which could have been derived from ibn Badr’s or some similar algebra. The same can be said about the give-and-take problems. So, the circumstantial evidence for an Iberian presence of ibn Badr’s work that follows from by manuscript location and hand is bolstered by mathematical contents.

Further evidence that Iberian *ḥisāb* had integrated algebra comes from Italy. As mentioned, Fibonacci had used Castilian material for his chapter on barter in the *Liber abbaci*, and the earliest extant abbacus text, the “Columbia Algorism” [ed. Vogel 1977], written around 1285–90 (though known from a fourteenth-century copy), has clear affinities to the Iberian tradition.[20] None of this, however, has to do with algebra (algebra is certainly to be found in the *Liber abbaci*, but in a different chapter). What concerns us is the earliest abbacus *algebra*, which is contained in the Vatican manuscript of Jacopo da Firenze’s *Tractatus algorismi* from 1307[21] – and what interest us in particular are the ten examples that illustrate the six basic cases. Five of these are formulated as pure-number problems (three about the “divided 10”, two about numbers in given proportion); the five others are supra-utilitarian *muʿāmalāt*-problems, dealing with a partnership;

19 In the first problem on p. 169, “el otro día antia 3” is a writing error for “el otro día antia 2”, either in the manuscript or in the edition.

20 For instance, it mostly presents rule-of-three solutions in the counterfactual form, “if so much were so much, ...”, which was standard in the Ibero-Provençal area but not done in later Italian abbacus books; cf. also [Høyrup 2005: 31 n.10, 42 n. 32]. For the dating, see [Høyrup 2007: 31 n. 70].

21 Discussion [Høyrup 2007: 100–121], edition pp. 304–331. Since the algebra chapter is contained in the Vatican manuscript only and absent from the Florence and Genova manuscripts, it could in principle be a secondary insertion. However, there are good reasons to assume that the Vatican manuscript represents the original faithfully, while the two other manuscripts (closely related to each other) derive from an adapted version – the algebra chapter, among other things, shares stylistic peculiarities with the other chapters in a way a secondary insertion would not do (cf. [Høyrup 2007: 6–25]). Be that as it may, the relation to other extant abbacus algebras shows that it precedes all of these, and also precedes 1328 by years. For simplicity, and since Jacopo is in any case known only as a name, I shall therefore refer to it as “Jacopo’s algebra”.

a loan; a give-and-take situation; profits on travels; and changing of Florentine into Venetian coin (and back). The give-and-take problem involves the square root of one of the possessions.

Jacopo wrote in Montpellier, and there are many reasons to believe that he took much of his material, and in particular the algebra, not from Montpellier itself but from somewhere in the Ibero-Provençal area – see [Høyrup 2007: 168*f*]. There are no Arabisms in his work, as there always are in first-generation medieval translations made from the Arabic. *Māl* ("possession"), moreover, is translated *censo*, in agreement with what had been the standard (as *census*) since 12th-century Iberian translations into Latin, but which has no sense in Italian. Jacopo's source tradition must thus already have worked in a Romance language. Actually, Fibonacci offers us evidence that algebra circulated in a Romance vernacular as early as the later twelfth or the early thirteenth century (and since algebra was unknown in Italy, as can be seen from the earliest abbacus texts, *where* if not in the Ibero-Provencal area?): when needing names for two unknowns in the *Flos* [ed. Boncompagni 1862: 236], he calls the second *res*, the regular translation of Arabic *šayʾ* used technically since Robert of Chester's translation of al-Khwārizmī's algebra. For the first unknown he chooses *causa*, coinciding with the medieval Catalan spelling of "thing" [Costa Clos & Tarrés Fernández 1998: 41] but not meaning "thing" in Latin.[22]

All in all it is thus a reasonable assumption that algebra was integrated in al-Andalus *muʿāmalāt*, and that even quite artificial problems involving square roots of real money were considered there. Since such problems are also found in the *Liber mahamaleth*, it seems an obvious conclusion that even this component of the book renders what the compiler-author found

[22] It does have this meaning in Merovingian, Lombard and Carolingian Latin [Niermeyer 1976: 159*f*; Du Cange et al 1883: II, 240*f*]. This explains the origin of the Catalan meaning as well as French *chose*, Italian *cosa* etc., which must all come from vulgar Latin; but it does not appear to have been used in literate medieval Latin in this sense elsewhere, in particular not where Fibonacci learned his Latin (elsewhere, in fact, he only uses the word with the meaning "cause").

Castilian *cosa* was also used for "thing" in the later Middle Ages – thus in the *Liber ... dicho alguarismo* – and could of course have been re-latinized by Fibonacci. The coinciding spellings are no proof that he took the word precisely from Catalan.

in the Arabic tradition. This is also what I believed myself until I read the book thoroughly for reviewing.

Algebra and proportion theory in the *Liber mahamaleth*

In order to see that it is a precipitate conclusion we will have to inspect a sequence of problems in detail. As an example we may look at a sequence about "buying and selling" [ed. Vlasschaert 2010: $_{\mathrm{II}}$193–211], following after the basic problems about this matter that are solved by the rule of three and its alternatives (those where division precedes multiplication). Using p and P for prices, q and Q for the appurtenant quantities, we have $\frac{q}{p} :: \frac{Q}{P}$ (it is to be observed that this is a proportion, not an equation involving two fractions[23]). The beginning of each problem will be indicated by page$_{\mathrm{line}}$ (omitting the initial subscript $_{\mathrm{II}}$ since everything regards the text edition).

193_7 $\frac{3}{13} :: \frac{Q}{P}$, $Q+P = 60$. This is solved by means of proportion theory, namely via transformation into $\frac{3}{3+13} :: \frac{Q}{Q+P}$ and subsequent use of the rule of three.

193_{32} $\frac{3}{13} :: \frac{Q}{P}$, $P–Q = 60$. This is solved analogously by being transformed into $\frac{3}{13-3} :: \frac{Q}{P-Q}$.

194_{13} $\frac{3}{8} :: \frac{Q}{P}$, $Q \cdot P = 216$. Nothing is said about fractions, but the rule given builds on awareness that

$$(3 \cdot 216) \div 8 = \tfrac{3}{8} \cdot 216 = \tfrac{Q}{P} \cdot (Q \cdot P) = Q^2$$

and

$$(8 \cdot 216) \div 3 = \tfrac{8}{3} \cdot 216 = \tfrac{P}{Q} \cdot (Q \cdot P) = P^2$$

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23 Strictly speaking, such expressions can of course not be proportions in the classical sense, since quantities and prices have different dimensions – but the text not only handles them as if they were but also states explicitly in the beginning of the chapter on buying and selling (p. $_{\mathrm{II}}$186) that the *proportio* of the first quantity to its price is as that of the second quantity to its price.

This may be elementary algebra in *our* sense but was certainly not understood as algebra in the twelfth century. In the end it is told what to do if the outcome of the calculation has no square root (namely to use the formula for approximation).

194_{27} $\frac{4}{9}::\frac{Q}{P}$, $\sqrt{Q}+\sqrt{P} = 7\frac{1}{2}$. It is used (but not said) that $\frac{\sqrt{4}}{\sqrt{9}}::\frac{\sqrt{Q}}{\sqrt{P}}$, which is no standard theorem from the theory of proportions[24] but follows easily from an arithmetical understanding. From here one proceeds as at 193_7.

An alternative presupposes that

$$\sqrt{\frac{4}{9}}+1 = \frac{\sqrt{Q}}{\sqrt{P}}+1 = \frac{\sqrt{Q}+\sqrt{P}}{\sqrt{P}} = \frac{7\frac{1}{2}}{\sqrt{P}} ,$$

which also points to an underlying arithmetical conceptualization.

Yet another alternative makes the claim that

$$\left(\sqrt{\frac{(\sqrt{P}+\sqrt{Q})^2}{(P-Q)\,/\,Q}+\left(\frac{\sqrt{P}+\sqrt{Q}}{(P-Q)\,/\,Q}\right)^2}-\frac{\sqrt{P}+\sqrt{Q}}{(P-Q)\,/\,Q}\right)^2 = Q ,$$

which is indeed true but not at all easy to verify without the modern symbolism in which I expressed the calculation (even with that tool at hand care is needed). No argument is given in the text.

195_{17} $\frac{4}{9}::\frac{Q}{P}$, $\sqrt{P}-\sqrt{Q} = 1\frac{1}{2}$. The three analogous procedures are prescribed.

196_1 $\frac{4}{9}::\frac{Q}{P}$, $\sqrt{Q}\cdot\sqrt{P} = 24$. Once again it is used (but not said) that $\frac{\sqrt{4}}{\sqrt{9}}::\frac{\sqrt{Q}}{\sqrt{P}}$. The problem is thus analogous to the one at 194_{13}. However, the first solution that is offered is

$$\frac{\sqrt{P}\cdot\sqrt{Q}}{\sqrt{4}\cdot\sqrt{9}}\cdot 4 = Q , \quad \frac{\sqrt{P}\cdot\sqrt{Q}}{\sqrt{4}\cdot\sqrt{9}}\cdot 9 = P ,$$

which suggests awareness that the initial proportion means that $Q = 4s$, $P = 9s$ with some shared factor s. We shall encounter the same insight below (problem at 201_{10}), where an explicit geometric argument is given

Alternatively, a procedure related to that at 194_{13} is suggested.

Finally, it is proposed to multiply 24 by itself, which yields

[24] The reverse, when formulated as dealing with the composition of ratios, certainly is.

PQ. Thereby, as pointed out, the problem becomes *strictly* analogous to that at 194_{13}.

Now a chapter follows "about the same, with [algebraic] things". I shall use *r* for *res* and *C* for *census* in my symbolic translations when these occur in the text.

196_{14} $\frac{3}{10+r}::\frac{1}{r}$.[25] This is transformed into $\frac{3}{10+r}::\frac{3}{3r}$, whence $3r = 10+r$, which is solved in the usual way. Alternatively, the proportion is transformed into $\frac{3-1}{(10+r)-r}::\frac{1}{r}$, that is, $\frac{2}{10}::\frac{1}{r}$, whence $\frac{1}{5}::\frac{1}{r}$, etc. As we see, cross-multiplication is not used to establish the equation; instead the antecedents are made equal, whence the consequents also become equal. This preference is general.

196_{26} $\frac{4}{20+2r}::\frac{1½}{2r+3}$. Through multiplication of the right-hand terms by $4\div1\frac{1}{2} = 2\frac{1}{3}$, this is transformed into $\frac{4}{20+2r}::\frac{4}{5\frac{1}{3}r+8}$, whence $5\frac{1}{3}r+3 = 20+r$, etc.

Alternatively: $\frac{4}{20+2r}::\frac{1½}{2r+3}::\frac{2½}{17}$. But $1\frac{1}{2}\div2\frac{1}{2} = \frac{3}{5}$, whence $\frac{1½}{2r+3}::\frac{1\frac{1}{2}}{\frac{3}{5}\cdot17}$, etc. It is pointed out that this ruse only works because we have the same multiple of *r* left and right.

197_{15} $\frac{8}{20+r}::\frac{2}{r-1}$. Transforming we find $\frac{2}{r-1}::\frac{6}{21}::\frac{2}{7}$, whence $r-1 = 7$, etc.

Alternatively, since $8\div2 = 4$, $\frac{8}{20+r}::\frac{8}{4r-4}$, etc.

197_{33} $\frac{6}{10+r}::\frac{2}{r}$. By transformation this yields $\frac{2}{r}::\frac{4}{10}::\frac{2}{5}$, whence $r = 5$.

198_{4} $\frac{6}{10+r}::\frac{2}{r+1}$. By transformation $\frac{2}{r+1}::\frac{4}{9}$, etc.

198_{14} $\frac{3}{20+r}::\frac{\frac{q}{2}}{\frac{2}{3}r-2}$, which is transformed into $\frac{3}{20+r}::\frac{3}{4r-12}$, etc.

198_{24} $\frac{6}{10-r}::\frac{2}{r}$. By transformation $\frac{2}{r}::\frac{8}{10}$, whence $\frac{8}{4r}::\frac{8}{10}$, etc.

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[25] In words: "Three measures are given for 10 coins and a thing, but this thing is the price of one measure". Similarly in the following questions.

An alternative that does not depend on the presence of precisely *one* thing left and right transforms the proportion into $\frac{6}{10-r} :: \frac{6}{3r}$, etc.

198_{34} $\frac{4}{8-r} :: \frac{2}{r+1}$. First solved via transformation into $\frac{6}{9} :: \frac{2}{r+1}$, which should give $\frac{2}{3} :: \frac{2}{r+1}$ but by error becomes $\frac{2}{3} :: \frac{2}{r}$, whence $r = 3$. Then, as in the previous example, by the more generally valid alternative, which gives the correct result $r = 2$. The discrepancy is not discussed and thus probably not noticed.

199_{11} $\frac{4}{20-2r} :: \frac{1½}{2r-3}$. Solved by the "general" method of the previous two examples. In the end it is pointed out that this can only be understood if one has studied algebra or Euclid's book, "which however have been sufficiently explained".

Next comes "another chapter about an unknown in buying and selling"

199_{27} An unknown number of measures is sold for 93, and addition of this number to the price of one measure gives 34 – in our symbols (since no *res* occurs): $x+\frac{93}{x} = 34$. At first the solution is given as $\frac{34}{2} \pm \sqrt{(\frac{34}{2})^2-93}$, the sign depending on whether the number of measures exceeds or falls short of the price of one measure. Next a geometric argument based on the principles of *Elements* II.5 is given. Since Euclid is not mentioned, which the compiler-author is elsewhere fond of doing, and since the argument uses a subdivided line only, the direct inspiration might be Abū Kāmil's similar proof for the fifth *al-jabr* case (possession plus number equals things) [ed. Sesiano 1993: 362*f*; ed. trans. Levey 1966: 88–90].

200_{27} The first of the two corresponding subtractive variants, namely the one in which the number of measures subtracted from the price of one of them gives 28. First a numerical prescription is given, next a line-based geometric proof. If instead (the second subtractive variant) it is subtraction of the number of measures from the price of one of them that gives 28, we are told to proceed correspondingly.

201_{10} $\frac{q}{p}::\frac{Q}{P}$, $pq = 6$, $PQ = 24$, $(p+q)+(P+Q) = 15$. Once again the argument appears to go via the factor of proportionality s, $sp = P$, $sq = Q$ (the ensuing geometric demonstration confirms this interpretation, cf. presently). At first s is found as $\sqrt{\frac{PQ}{pq}} = 2$. Therefore $p+q = \frac{1}{1+2}\cdot(p+q+P+Q) = \frac{1}{1+2}\cdot 15 = 5$. Since we already know pq, we can proceed according to the fifth case of *al-jabr* or *Elements* II.5, none of which are mentioned; the double solution *is*, however. A geometric argument explains the finding of the factor of proportionality; for the last part of the demonstration, a cross-reference to the problem at 199_{27} is deemed sufficient.

202_{34} $\frac{q}{p}::\frac{Q}{P}$, $pq = 10$, $PQ = 30$, $(p+q)+(P+Q) = 20$. This seeming exact analogue of the preceding question leads to an irrational value $s = \sqrt{3}$, and therefore to complications and a cross-reference to the chapter about roots (where indeed the necessary explanations are found). In the end, this leads to a discussion in terms of the classification of *Elements* X (not mentioned here, which suggests that these classes are supposed to be familiar – elsewhere the book *is* mentioned).

204_{24} $\frac{q}{p}::\frac{Q}{P}$, $pq = 6$, $PQ = 24$, $(P+Q)–(p+q) = 5$. The first part of this subtractive variant of the problem at 201_{10} is a prescription analogous to the one of the additive variants; for the second part, a mere cross-reference is given.

204_{35} $\frac{q}{p}::\frac{Q}{P}$, $pq = 6$, $PQ = 24$, $(p+q)\cdot(P+Q) = 10$. Without being identified, the proportionality factor s is found as $\sqrt{\frac{PQ}{pq}}$; next (since $P+Q = s(p+q)$, which is not explained) $p+q$ is found as $\sqrt{\frac{(p+q)\cdot(P+Q)}{s}}$. For the rest, a cross-reference is given. For the first step, however, a geometric demonstration is supplied in the end.

205_{21} $\frac{q}{p} :: \frac{Q}{P}$, $pq = 20$, $PQ = 10$, $(p+q)\cdot(P+Q) = \sqrt{5760}$. This is explained to follow the previous question, but of course gives rise to complicated manipulations of roots, for which reason both ways to solve the problem are discussed in detail.

207_{2} $\sqrt{p} = 3q$, $p–q = 34$ (the identification of the two numbers as price and appurtenant quantity is quite dispensable). The solution follows from a quadratic completion ($\sqrt{p} = t$, $q = \frac{1}{3}t$):

$$t^2 - \frac{1}{3}t = 34$$

$$t^2 - 2\cdot\frac{1}{6}t + \left(\frac{1}{6}\right)^2 = 34\frac{1}{36}$$

$$t - \frac{1}{6} = 5\frac{5}{6}$$

$$t = 6$$

At first a purely numerical prescription is given, afterwards follows a geometric, line-based proof.

207_{24} $\sqrt{p} = 2q$, $p+q = 18$. *Mutatis mutandis*, this additive counterpart of the preceding problem is solved in the same way, and similarly provided with a line-based geometric proof.

208_{9} $\frac{6}{4+r} :: \frac{2}{3\sqrt{(4+r)}}$, transformed into $\frac{6}{4+r} :: \frac{6}{9\sqrt{(4+r)}}$. The resulting equation $(4+r) = 9\sqrt{4+r}$ is not made explicit, but the numerical prescription corresponds to its transformation into $\sqrt{4+r} = 9$ and further into $4+r = 81$.

208_{15} $\frac{6}{4–r} :: \frac{2}{3\sqrt{(4–r)}}$. Solved correspondingly.

208_{21} $\frac{3}{x+y} :: \frac{1}{y+\frac{1}{9}y}$, $xy = 21$ (x and y stand for what is spoken of as “two different things”). A prescription is given which corresponds to the transformation of the proportion into $\frac{3}{x+y} :: \frac{3}{3y+\frac{1}{3}y}$, whence $x+y = 3\frac{1}{3}y$, $x = 2\frac{1}{3}y$, $2\frac{1}{3}y^2 = 21$, $y^2 = 9$, and finally $y = 3$, $x = 7$. After the prescription comes a line-based argument corresponding to the symbolic equations.

Alternatively, the problem can be solved "according to algebra". Here, the *thing* (r) takes the place of y, while the *dragma* (d) takes that of x. This time, the equation comes from a different but similar transformation of the proportion, namely into $\frac{1}{\frac{1}{3}d+\frac{1}{3}r}::\frac{1}{r+\frac{1}{9}r}$. From here follows the equation $r+\frac{1}{9}r = \frac{1}{3}d+\frac{1}{3}\mathrm{r}$, whence $d = 2\frac{1}{3}r$. Inserting this in $rd = 21$ we get $2\frac{1}{3}C = 21$, $C = 9$, $r = 3$.

209_{30} $\frac{5}{x+y}::\frac{1}{\frac{1}{3}x+2}$, $xy = 144$. Both methods of the previous problem are applied, now evidently leading to mixed second-degree problems; the line-argument goes through the complete calculation, whereas the algebraic solution, once it is found that $d = \frac{2}{3}r+10$, merely says that "the rest is done as we have taught in the algebra".

For other classical *muʿāmalāt* topics (profit and interest, partnership, etc.), we find similar systematically varied problem sequences, involving for example sum, difference and product of capital and profit, often constructed so as to fit the application of proportion theory. Obviously, we are far from the habitual presentation of select "rarities" within or outside specific *muʿāmalāt* chapters of *ḥisāb books* and closer to theoretical exploration. The supra-utilitarian rarities of the *ḥisāb* tradition with their occasional references to such impossible things as the square root of a dowry have suggested a new and unaccustomed, "*muʿāmalāt*-like" domain that could be submitted to quasi-theoretical scrutiny. But what is done in the *Liber mahamaleth* is certainly new relatively to the *muʿāmalāt* tradition.

Could this be Arabic?

One question remains: is this novelty due to the Latin compiler-author, or is he using or inspired by an Arabic treatise where something similar was already taking place?

It is hardly possible to give a definitive answer to this question, but several sources speak strongly in favour of the latter possibility when considered in combination.

Firstly, one source exists which shows that the existence of an Arabic precursor is at least not to be ruled out. This source is the first, little studied part of chapter 15 of the *Liber abbaci*, treating purportedly "the proportions of three or four quantities, to which the solution of many questions belonging to geometry are reduced" [ed. Boncompagni 1857a: 387]. Actually, it deals with *numbers* in proportion – see the analysis in [Høyrup 2011: 89–92, 97–100].

Fibonacci starts by considering questions involving three numbers. In paragraphs (1)–(3), these numbers are in continued proportion. One of the numbers is given together with the sum of the other two. Line segments are named in the alphabetic order *a*, *b*, *c*, ... Paragraphs (4)–(38) also deal with three numbers, but now differences between the numbers are among the magnitudes considered. The alphabetic order changes to *a*, *b*, *g*, *d*, ..., yet paragraphs (4)–(5) still make use of the letter *c* in manipulations, and manipulations as well as the ensuing observation (6) sometimes designate a segment by a single instead of two letters; none of the following paragraphs do so. Finally, paragraphs (39)–(50) consider four numbers in proportion. The underlying alphabetic order is still *a*, *b*, *g*, *d*, ...

The change of alphabetic order leaves no doubt that Fibonacci has used sources in Greek or Arabic tradition for paragraphs (7)–(50) – the hybrid system of (4)–(6) suggests, either that he has produced these questions himself but tried to emulate what follows, or (rather) that he has modified the procedures of borrowed material in these questions.

This only shows its relevance to our present concern if we look more closely at the text; we may restrict our interest to the sequence (7)–(38).

This sequence can be described as a systematic investigation of the properties of the different "means" Q between two numbers P and R, $P<Q<R$, known from Nicomachos's *Introduction to Arithmetic* (omitting the arithmetical mean, which is trivial, and inserting $\frac{R-P}{R-Q} :: \frac{R}{Q}$, which has been left out by Nicomachos but is present in Pappos's similar list)[26] – namely

26 For these means, cf. [Heath 1921: II, 85–88]. Details in the formulation show that Nicomachos, not Pappus (from whose list a different mean is lacking) has provided the starting point.

showing for each of them how any of the three numbers can be determined if the other two are known.

As for example pp. 193–211 in the *Liber mahamaleth*, this sequence thus presents us with a systematic theoretical scrutiny of an existing mathematical domain under a new perspective – here Nicomachos's list of means, which in itself is no more theoretical than the *muʿāmalāt* tradition (except of course in the Greek sense of being decoupled from any external practice). And as in the *Liber mahamaleth*, the tools are proportion theory and *Elements* II.5–6.

Fibonacci's source for this first part of Chapter 15 *could* be an unknown Greek treatise; more likely it was Arabic, from al-Andalus or from elsewhere. In any case he shows us that *somewhere* in the region from which he drew his inspiration he had encountered concerns similar to those that are in evidence on pp. $_{\text{II}}$193–211 in the *Liber mahamaleth*. In consequence it is not to be excluded that the compiler-author of the latter work could have drawn on something similar. The second part of the chapter, concerned purportedly with "geometrical questions", offers som positive evidence. A number of these questions are indeed not geometrical at all but deal with composite gain on travels – see [Høyrup 2011: 92*f*]. This lead to questions which, expressed in equations, would be of the second degree, one of them mixed [ed. Boncompagni 1857a: 399]. It is not solved by means of algebra, however, but through manipulation of proportions and a line argument (with alphabetic order *a*, *b*, *g*, ...) corresponding to but not explicitly referring to *Elements* II.6 – strikingly similar to what is often done in the *Liber mahamaleth*.

At this point we may return to Gundisalvi's *De divisione philosophiae* and his reference to "the book which in Arabic is called *Mahamalech*". That could well be a book, not about *muʿāmalāt* simply but about *muʿāmalāt vom höheren Standpunkt aus*, "from a higher vantage point", containing both the systematic algebraic and proportion-theoretical expansion which we have just discussed and the many demonstrations in Euclidean style – and even something like the metamathematical introduction. Translation from such a book would eliminate the main objection to the ascription of the *Liber mahamaleth* to Gundisalvi: namely that nothing in the list of his writings in [Kren 1972] suggests a *working competence* in mathematics at the level the *Liber mahamaleth* bears witness of.

That the *Liber mahamaleth* should be seen as a (possibly more or less free) translation of the "book called *Mahamalech*" is strongly supported by comparison of (1) what Gundisalvi says about the topics dealt with in the latter [ed. Baur 1903: 93] with (2) what is said in the introduction to the *Liber mahamaleth* about its contents [ed. Vlasschaert 2010: $_{\mathrm{II}}$7] – quoted in Latin with italics added in order to show the almost perfect agreement:

(1) sciencia *uendendi et emendi, alia mutuandi et accommodandi; alia est conducendi et locandi; alia est expendendi et conseruandi.*

(2) Scientia uero negociandi: alia est *uendendi et emendi, alia* est *mutuandi et accomodandi, alia est conducendi et locandi, alia expendendi et conseruandi, et multe alie de quibus in sequentibus tractabitur.*

These observations entail that the "compiler-author" should almost certainly be split into two persons: an Arabic author (probably an astronomer-mathematician, given the fondness of proportion techniques) and a Latin compiler-translator who is either Gundisalvi himself or one of his close collaborators.[27] This split, by the way, opens the possibility that the missing algebra chapter was lost already in translation, since "the original version of the *Liber mahamaleth* [which] contained an orderly presentation of algebra" might well be the *Liber mahamalech*.[28]

The same observations do not imply that everything in the *Liber mahamaleth* is taken over from the *muʿāmalāt* tradition; it is not. They only indicate with high probability that the *Liber mahamaleth* is taken over from an Arabic book that had borrowed the *muʿāmalāt* title but already integrated material which was foreign to *al-muʿāmalāt* and submitted

27 An argument against the alternative ascription to Johannes Hispalensis is the algebraic terminology. The "Toledan *regule*" contain a small "excerpt of the book called *gebla mucabala* [ed. Burnett, Zhao & Lampe 2007: 163–165], where *res* is used as translation of *māl*, which in the *Liber mahamaleth* is always *census* (since the excerpt covers only the basic cases with their standard examples, there is no occasion to translate *šayʾ*, the *res* of the *Liber mahamaleth*).

28 We may also observe that the constant alphabetic order of the geometric proofs (line proofs and otherwise) is *a, b, g, d, ...* – but this would evidently also be the case if a Latin compiler had translated from a plurality of Arabic texts.

everything to a theoretical perspective – treating it *magistraliter*, as Fibonacci [ed. Boncompagni 1857: 163, 215, 364] was going to say about his similar endeavour in the book whose title *Liber abbaci*[29] is equally misleading by (false or genuine?) modesty.

An addendum about the Indian summer of al-Andalus mathematics

This was as much as I have to say about the *Liber mahamaleth,* but there is something of a more general nature to add. The starting point for this is another historical detail, namely the problem type known as "the unknown heritage". In the following I epitomize some results of an investigation of its temporal and geographical distribution which I published in [Høyrup 2008].

In the standard version of this problem it states that a father leaves to his first son 1 monetary unit and $\frac{1}{n}$ (n usually being 7 or 10) of what remains, to the second 2 units and $\frac{1}{n}$ of what remains, etc. In the end all sons get the same amount, and nothing remains. The solution is that there are n–1 sons, each of whom receives n–1 monetary units. In a variant of this simple version, the fraction is given first and the arithmetically increasing amount afterwards.

But there are also "sophisticated" versions in which n is not integer, and in which a different arithmetical series intervenes.[30] If we take for granted that there *is* a solution, that of the simple versions can be found directly, without the use of algebra or other advanced methods, from the equality of the last two shares; but that has escaped everybody, also modern commentators. No similar solution exists for the sophisticated versions. Some medieval versions use the equality of the first two shares

[29] That is, one of the titles. *Liber abbaci* is how Fibonacci refers to it in the *Pratica geometrie* [ed. Boncompagni 1862: 9, 24, 81, 148]; on p. 9 it also appears as *Liber maioris guice abbaci*. In the dedicatory letter to the *Flos* it is *Liber maior de numero,* and later simply *Liber de numero* (*id.* p. 227, 234), as also in the letter to Theodorus (*id.* p. 247) and the *Liber quadratorum* (*id.* p. 253)

[30] Since arithmetically increasing decreases would result in a parabolic descent, we may speak of "parabolic-geometric series".

to solve the simple problem by means of algebra or the double false position.

Not only the simple but also the sophisticated types are found in the *Liber abbaci*, which is our earliest source for both. Fibonacci gives an algebraic solution to one of the latter. He also gives rules for solution of all of them, which however are *not* derived from his algebra and must hence be borrowed from some earlier work. This earlier work must have had the character of a theoretical analysis of the conditions that a parabolic-geometric series[31] be also arithmetical (expressed of course in wholly different terms).

Both types appear to be completely absent from known Arabic texts, but ibn al-Yāsamīn's *Talqīḥ al-afkār fī'l ʿamali bi rušūm al-ghubār* (written in Marrakesh in c. 1190) contains a slightly simplified and clearly secondary version of the simple problem.[32]

The simple versions turn up in many Italian abbacus books (since 2008 I have found a number of supplementary occurrences, which however do not change the picture), and in a pure-number version in Maximos Planudes's late thirteenth-century *Calculus According to the Indians, Called the Great* [ed., trans. Allard 1981: 191–194]. It is also present in the *Libro ... dicho alguarismo* (twice p. 169), which I had overlooked and explicitly denied in [2008: 632]. The sophisticated versions turn up again in Barthélemy de Romans' *Compendy de la praticque des nombres* [ed. Spiesser 2003: 26, 30].[33] Here, only rules, no argument for their validity (except a mock solution by double false) are given; these rules are not identical with those of Fibonacci but similar in kind.

It is possible to find by means of symbolic algebra the rules for the sophisticated versions and to show that they actually make all shares equal. But line-based proofs as we know them from the *Liber abbaci* and the *Liber*

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31 That is, a series with a double decrease, of which the first in isolation would produce a parabolic and the second alone a geometric descent.

32 Another simplified version is found in the *al-Maʿūna fī ʿilm al-ḥisāb al-hawāʾī* written by Ibn al-Hāʾim (1352–1412, Cairo, Mecca & Jerusalem, and familiar with Ibn al-Yāsamīn's work).

33 Probably written around 1467 but only known from a revised redaction which was prepared by Mathieu Préhoude in 1476.

mahamaleth will do just as well.

Putting together the complete evidence I concluded in [2008] that the simple versions have probably originated in the Greek world in late Antiquity or in the Byzantine Middle Ages. The sophisticated versions and the way to find rules for solving them, I inferred, must in all likelihood be traced to Provence or al-Andalus. Given that the simple version was actually known in the Iberian world and that nothing suggests the presence of mathematicians at or above Fibonacci's level in Provence before Fibonacci's time, al-Andalus is more likely than Provence.

So, it seems that al-Andalus produced, in part before c. 1160, in part at least before 1200, the *Liber mahamaleth*; a systematic investigation of the Nicomachean means; and an investigation of parabolic-geometric series. This fits what was said by Djebbar in [1993: 86], namely that there was

> in Spain and before the eleventh century, a solid research tradition in arithmetic whose starting point seems to have been the translation made by Thābit ibn Qurra of Nicomachos' *Introduction to Arithmetic*.

So far we have no Arabic testimonials that confirm our conclusions, only texts in Latin and Romance vernacular. This, however, agrees with the general picture of the fate of the erudition belonging to the Indian summer of al-Andalus. Al-Muʾtaman's eleventh-century *Kitāb al-Istikmāl* still gave rise to further work not only by Maimonides but also by Arabic mathematicians [Djebbar 1993: 82 and *passim*]; but Jābir ibn Aflaḥ's twelfth-century work is much better known from Hebrew and Latin translations than in Arabic [Lorch 1973: 39]. Even ibn Rušd, as known, had scant influence in the Arabic world [Arnaldez 1969: 919] but very much on Hebrew and Latin philosophy. It looks as if the "book which in Arabic is called *Mahamalech*", the close scrutiny of the means, and the examination of the properties of parabolic-geometric series shared his fate.

Indian summers are wonderful as long as they last. But they do not last – and the blossoms they produce may never get the time to ripen into fruit.

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